Signed skeletons of three-valent polyhedra: the catalogue to eight vertices and the smallest nonplanar skeleton

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Abstract

A three-valent polyhedral solid (a "3VP" in Knuth's terminology) has exactly three faces at every vertex; its signed skeleton is its edge graph with each edge marked convex or concave. Knuth asked which signed cubic graphs arise this way, gave a catalogue for graphs with at most eight vertices, and asked for rigorous impossibility proofs for $K_{3,3}$ and for one signature of the triangular prism. We prove six elementary lemmas about the local and global geometry of 3VPs and use them, with a computer enumeration of sign patterns over all sphere face structures, to determine the realizable signatures on at most eight vertices completely: there are exactly thirty. Twenty-nine are in Knuth's catalogue; the thirtieth is a new signature of the lopped triangular prism, for which we give integer coordinates and an exact rational certificate. We also answer Knuth's question about the smallest nonplanar skeleton: it has exactly fourteen vertices, as he conjectured, attained by the Szilassi polyhedron. The proof rests on a parity lemma for the vertices on the line where two face planes meet, which is linear over $\mathbf{F}_2$ and independent of the arrangement, so that the ten- and twelve-vertex cases reduce to finite combinatorial searches supported by one exhaustive search over the 43 combinatorial types of six planes in general position, carried out in exact rational arithmetic.

1 Introduction

In a note dedicated to the memory of John Conway, Knuth [1] considers compact polyhedral solids in which exactly three faces meet at every vertex, calls them 3VPs, and marks each edge of such a solid *convex* or *concave* according to whether the interior dihedral angle is less or greater than 180°. The resulting cubic graph with signed edges is the *signature* of the solid, and Knuth asks which signed cubic graphs are signatures. He gives a hand-built catalogue of realizable signatures for the cubic graphs with at most eight vertices (Read and Wilson's [14] C1 through C8 and $K_4 \oplus K_4$), lists a third potential signature of the prism C2 and the graph C3 = $K_{3,3}$ as "surely unrealizable" while noting that he knows no rigorous proof, and asks whether his catalogue is complete.

We settle these questions for at most eight vertices.

Theorem 1. *Exactly thirty signed cubic graphs with at most eight vertices are signatures of 3VPs: one for K_4, two for the prism C2, fifteen for the lopped prism C4, two for C6, six for*

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the cube C8, and four for $K_4 \oplus K_4$. The graphs $K_{3,3}$, C5 and C7 have no realizable signature. Knuth's catalogue contains twenty-nine of the thirty; the missing one is a signature of C4 with three concave edges forming a path (Figure 2).

The proof combines six lemmas (Section 3) with an enumeration (Section 4); further lemmas settle Knuth's question about the smallest nonplanar skeleton: it has fourteen vertices, as he conjectured (Section 7). The lemmas are elementary: a classification of the local solid at a vertex, which is essentially the trihedral junction catalogue of Huffman [2] and Clowes [3]; a convex-hull argument; a counting argument on face planes; the converse of Steinitz's theorem; a half-space lemma relating the sign of an edge to the side of the face plane on which the neighbouring face lies; and a lemma about two faces that share two edges. The enumeration runs over all rotation systems of the cubic graphs on at most eight vertices in which every component is a sphere, and over all 2^E sign patterns, discarding patterns that violate a lemma. Every survivor is realized, either by one of Knuth's published coordinate models or by the new realization, and all realizations are certified in exact rational arithmetic (Section 5). Faces with holes and disconnected skeletons are handled by hand (Section 6).

Related work. Realizability of *labelled line drawings*, in which a two-dimensional projection is given together with convex/concave/occluding labels, was reduced to linear programming by Sugihara [4]; Whiteley [5] treated realizability of polyhedra by projective methods. Knuth's question has no projection, so those results do not apply directly. Eppstein and Mumford [6] characterized the graphs of simple orthogonal polyhedra, an all-right-angle subclass of 3VPs.

2 Definitions

Following [1], a *3VP* is a compact set $X \subset \mathbb{R}^3$ with nonempty interior whose boundary ∂X is a finite union of planar polygons (*faces*), each a simple polygon with zero or more simple polygonal holes, such that every edge (a side of a face polygon or of a hole polygon) is a side of exactly two faces; every vertex lies on exactly three faces and three edges; adjacent faces (faces sharing an edge) are not coplanar; and near each interior point of a face, X lies on exactly one side of the face. X need not be connected and may contain cavities and tunnels. The *skeleton* is the cubic graph of vertices and edges. The *type* of a vertex is its number of concave edges.

Three facts are used throughout. (F1) ∂X is a closed 2-manifold, orientable because it bounds a compact region of $\mathbb{R}^3$: edges are two-sided by definition, and at a vertex the three faces form a single cycle. (F3) The three face planes at a vertex meet only at that vertex: each edge at v lies on the intersection line of two of them, and a common line would carry all three edges, but only two edge directions leave v along a line. (F2) No face has an interior angle of exactly 180° at a vertex: if face F had a straight angle at v, the two edges of F at v would be collinear, and each lies in one further face, so the line would lie in all three planes at v, which by F3 meet only at v.

3 Six lemmas

Lemma 2 (local structure; Huffman [2], Clowes [3])**.** *Let v be a vertex with face planes P_1, P_2, P_3. Near v the solid is one of: a convex trihedral cone K (type 0); $H \setminus W$ for a half-space H and a convex wedge W with v on the edge of W (type 1); the closure of the complement of such an*

$H \setminus W$ (type 2); or the closure of the complement of K (type 3). Consequently, at a vertex of type 1 the two convex edges lie in one face, whose interior angle at v exceeds $180°$, while the other two faces have angles below $180°$; at a vertex of type 2 the same holds with the two concave edges; at types 0 and 3 all three face angles are below $180°$.

Proof. Near v the boundary lies in $P_1 \cup P_2 \cup P_3$, and X is closed with $X \setminus \partial X$ open, so X is a union of closed cells of the arrangement. Identify the eight cells with the octants of the sphere of directions cut by three great circles; two octants are adjacent if they share an arc. A union U of octants is a 3VP vertex exactly when its boundary curve consists of three maximal great-circle arcs (consecutive arcs on one great circle merge into one planar sector). Complementation flips all signs, so it suffices to examine unions of $k \leq 4$ octants. $k = 1$: three arcs; X is the convex cone K. $k = 2$: two adjacent octants form a wedge (no vertex); two non-adjacent ones meet in a point or not at all, and the boundary is not a manifold at v. $k = 3$: the connected unions are paths $a \sim b \sim c$, and each is $\{z \leq 0\} \setminus \{x > 0, y > 0\}$ up to symmetry, whose boundary is a $270°$ sector in ∂H plus two quarter-planes: this is $H \setminus W$. $k = 4$: the four octants around a pole form a hemisphere (no corner); the path $(+++)(++-)(+--)(---)$ has four corners; an octant with its three neighbours has six; every other union is pinched or disconnected. The sign statements follow: for K all edges are convex; for $H \setminus W$ the edge of W is concave and the two edges in ∂H are convex, the face in ∂H has angle $360° - \theta > 180°$ where $\theta < 180°$ is the angle of the section of W by ∂H, and the two faces on ∂W have angles below $180°$; complements flip signs and keep angles. □

By Lemma 2, the reflex angles of a face are exactly the *demands*: face F has a reflex angle at v iff v has type 1 or 2 and F contains its two like-signed edges. A triangle carries no demand, and a simple k-gon carries at most $k - 3$ (its angles sum to $(k - 2) \cdot 180°$).

Lemma 3 (hull)**.** *Every 3VP has at least four vertices of type 0.*

Proof. $\mathrm{conv}(X)$ has at least four vertices since X is not planar. A vertex w of $\mathrm{conv}(X)$ is an extreme point of X, hence not in the relative interior of an edge or face, hence a vertex of X; and the tangent cone of $\mathrm{conv}(X)$ at w is pointed, so the local cone of X at w contains no line. Of the four cones in Lemma 2, $H \setminus W$ and its complement contain a line (for $\{z \leq 0\} \setminus \{x > 0, y > 0\}$, the x-axis), and the complement of K contains lines; only K is pointed. □

Lemma 4 (four planes)**.** *Every connected component S of ∂X lies in at least four planes, and if it lies in exactly four it is the boundary of a tetrahedron.*

Proof. Let R be the bounded component of $\mathbb{R}^3 \setminus S$ and Π the union of the planes of the faces on S. $R \cap (\mathbb{R}^3 \setminus \Pi)$ is open and closed in each cell of the arrangement of Π, so R is a union of cells. A nonempty polyhedron $\{x : a_i \cdot x < b_i\}$ is bounded iff its recession cone $\{d : a_i \cdot d \leq 0\ \forall i\}$ is $\{0\}$, iff the a_i positively span $\mathbb{R}^3$ (the recession cone is the polar of the cone generated by the a_i, and a closed convex cone is $\{0\}$ iff its polar is the whole space); a positively spanning set has at least four vectors. So cells of at most three planes are unbounded, and S lies in at least four planes. With exactly four, a bounded cell is cut out by all four half-spaces with outer normals n_i positively spanning, so any three are independent (no parallels, no three planes through a line); if the planes had a common point p the cell $\{x : n_i \cdot (x - p) < 0\}$ would be empty; hence the planes are in general position and the cell Q is a tetrahedron. It is the only bounded cell: with $\sum \lambda_i n_i = 0$, $\lambda_i > 0$, every dependency is a multiple of λ; a bounded cell $\{x : s_i(n_i \cdot x - d_i) < 0\}$ needs $\sum \mu_i s_i n_i = 0$ with $\mu_i > 0$, so $(\mu_i s_i) = t\lambda$ and all $s_i = \mathrm{sign}(t)$; the choice $s_i = +1$ is Q, and the reversed cell is empty because $\sum \lambda_i(n_i \cdot x - d_i)$ is a constant, negative on Q. So $R = Q$ and $S = \partial Q$. □

Corollary 5. *Let a connected component S of ∂X, of genus g, carry V vertices (its skeleton may be disconnected), F faces and H holes in total. Then $F = V/2 + 2 - 2g + H$; and if every face on S is a disc, the skeleton on S is connected, since otherwise the closed faces with boundary in one part of the skeleton and those with boundary in the other would be two disjoint closed sets covering S. In particular $K_{3,3}$ is unrealizable, every connected skeleton on at most eight vertices without holed faces lies on a sphere, and C5 and C7 are unrealizable (their holed-face structures are excluded in Section 6).*

Proof. Euler's formula with faces of Euler characteristic $1 - h$, which does not use connectivity of the skeleton. For $K_{3,3}$: a holed face needs two vertex-disjoint cycles, impossible with girth 4 on six vertices, so $H = 0$; nonplanarity gives $g \geq 1$ and $F \leq 3$, contradicting Lemma 4. For $V \leq 8$, $H = 0$, $g \geq 1$: $F \leq 4$, with equality only for $V = 8$, $g = 1$, $F = 4$, but four planes bound only a tetrahedron, a sphere. □

Lemma 6 (all-convex signatures)**.** *If every edge is convex, each component of X is convex and its skeleton is 3-connected.*

Proof. A compact connected locally convex set is convex (Tietze–Nakajima [8, 9]); local convexity holds at faces, at convex edges, and at type-0 vertices. Steinitz's theorem [7] gives 3-connectivity. A cavity would have all edges concave from the solid's side, so there is none. □

Lemma 7 (half-space; the side constraint of Sugihara [4])**.** *Let A be a face with outward normal u and F the other face at an edge e of A. Every vertex of F on the near side of e (the open half-plane of* $\mathrm{plane}(F)$ *bounded by* $\mathrm{line}(e)$ *that contains the interior of F near e) lies strictly outside* $\mathrm{plane}(A)$ *if e is concave and strictly inside if e is convex; vertices on the far side lie on the opposite side. If F is convex, all its vertices other than the ends of e are near. If F is a quadrilateral with one reflex vertex r (a dart), then for an edge (r, x) the vertex adjacent to x is near and the vertex adjacent to r is far, and for the other edges both remaining vertices are near.*

Proof. $\mathrm{plane}(F) \cap \mathrm{plane}(A) = \mathrm{line}(e)$, so each open half-plane of $\mathrm{plane}(F)$ bounded by $\mathrm{line}(e)$ lies on one side of $\mathrm{plane}(A)$. At an interior point of e the solid is the sector between the half-planes carrying A and F (convex) or its complement (concave), so the half-plane carrying F lies on the side $-u$ in the convex case and $+u$ in the concave case. A convex polygon has no third vertex on the line of an edge (it would have a straight angle, excluded by F2). A dart's reflex vertex r lies inside the triangle of the other three, and the dart is that triangle minus the triangle cut off by the two edges at r; the line through r and x separates the two remaining vertices, and the interior of the dart near (r, x) is on the side away from the cut-off triangle, i.e. the side of the vertex adjacent to x. Edges not at r are sides of the hull triangle. □

Lemma 8 (shared line)**.** *Let faces F and G share two edges e, e'. Then e, e' lie on the line $\ell = \mathrm{plane}(F) \cap \mathrm{plane}(G)$;* $\mathrm{sign}(e) = \mathrm{sign}(e')$ *iff e, e' are traversed in the same direction along ∂F (equivalently along ∂G); F and G are not convex; and in the co-directed case, if both boundary arcs of F between e and e' have three vertices (endpoints included), one arc has no reflex vertex and the other contains all reflex vertices of F.*

Proof. By Lemma 7, sign(e) is convex iff the near half-plane of G at e lies on the inner side of plane(F); the two half-planes of plane(G) bounded by ℓ lie on opposite sides of plane(F), and X is on one fixed side of plane(F) along F; the near half-planes at e and e' coincide iff the edges are co-directed along ∂G, and G traverses each edge oppositely to F. Walk ∂F (a disc face) with the interior on the left; exterior angles lie in $(-180°, 180°) \setminus \{0\}$, negative exactly at reflex vertices, and sum to 360°. The turning T_1 from the direction of e to that of e' is the sum of exterior angles at the head of e, the vertices between, and the tail of e'; T_2 is the sum over the other arc; $T_1 + T_2 = 360°$. Co-directed means $T_1 \equiv T_2 \equiv 0 \pmod{360°}$. An arc of m angles summing to $360°j$ has at least $2j+1$ positive angles; with $m = 3$ this forces $\{T_1, T_2\} = \{0°, 360°\}$, the 360° arc all convex and the 0° arc containing a reflex vertex. In the opposite case F has edges on ℓ pointing both ways, so its interior lies on both sides of ℓ and F is not convex; in the co-directed case the 0° arc has a reflex vertex. The same holds for G. □

In the enumeration below, the only pairs of faces sharing two edges are the two hexagons of C6, and there both arcs have three vertices.

4 The enumeration

For each cubic graph on $n \leq 8$ vertices we enumerate the rotation systems in which every component is a sphere (justified for connected skeletons without holed faces by Corollary 5), and for each resulting face structure all 2^E sign patterns, discarding a pattern if a type-1 or type-2 vertex places its demand on a triangle, if a k-gon receives more than $k-3$ demands (Lemma 2), if fewer than four vertices have type 0 (Lemma 3), if the half-space constraints of Lemma 7 for convex and dart neighbours conflict, or if the sign or arc conditions of Lemma 8 fail. Lemma 6 removes the all-convex pattern of C6. Two implementations with different data structures (by the same author, so a consistency check rather than an independent replication) agree on the survivors up to isomorphism of signed graphs:

$$K_4 : 1, \quad \text{C2} : 2, \quad K_{3,3} : 0, \quad \text{C4} : 15, \quad \text{C5}, \text{C7} : 0, \quad \text{C6} : 2, \quad \text{C8} : 6, \quad K_4 \oplus K_4 \text{ (disc faces)} : 2.$$

Every survivor is realized. We recomputed the signatures of Knuth's published coordinate models from his data file and matched them to the survivors as signed graphs: all of C1, C2, C2A, C4A1–C4C1, C6A, C6B, C8A–C8F appear, and the single unmatched survivor of C4 is realized by the coordinates of Figure 2. The two survivors of $K_4 \oplus K_4$ are the two convex tetrahedra and a tetrahedron with a tetrahedral cavity; the glued and dented tetrahedra of Knuth's catalogue have a holed face and are recovered in Section 6. Figure 1 shows all thirty signatures.

5 Certificates

A realization is certified by an exact rational computation that checks: every face is planar; every face polygon is simple and has no straight angle; the faces, oriented consistently, enclose positive volume; every edge has the claimed sign, computed as the sign of $n_A \cdot (n_B \times d)$ for the outward Newell normals of the two faces at the edge; no two faces meet except along a shared edge or at shared vertices (ear-clipping triangulation and exact segment–triangle tests); and, as an independent test, no skeleton edge meets any face except at shared vertices or as a shared edge. Knuth's coordinate models are given to a few decimals; we snapped each to exact rational face planes (each vertex being the intersection of its three planes) and certified the result. All 35 disc-faced models certify, including the Szilassi polyhedron. The new realization was found by a penalty-method search over vertex coordinates with the face structure fixed, then snapped to planes on the grid $\frac{1}{2}\mathbb{Z}$, which is why its coordinates are integers after scaling by 24.

6 Holed faces and disconnected skeletons

A face with a hole has an outer boundary cycle C_1 and a hole cycle C_2, vertex-disjoint, in the face plane P. Each of their vertices has two edges in P and a third edge that leaves P, since otherwise all three faces at that vertex would contain two edges in P and lie in P. With at most eight vertices this leaves at most two vertices off P, so $|C_1| = |C_2| = 3$, and the off-plane vertices x, y are adjacent only to $C_1 \cup C_2$, each vertex of which has exactly one of x, y as a neighbour. A face with two holes would need nine vertices, and two holed faces are impossible: their six-vertex boundaries share at least four vertices, which lie on the common line of two distinct planes (forcing three collinear vertices of a triangle or a hole vertex on the line of an outer side) or, if the planes coincide, put two coplanar adjacent faces at a shared vertex. A machine check confirms that only two configurations pass the degree count. (a) x sees all of C_1 and y all of C_2: the graph is $K_4 \oplus K_4$. The holed face has angle 360° minus the hole angle at each hole vertex, so by Lemma 2 the like-signed pair there is the two hole edges: the three hole edges share a sign and the three edges to y have the opposite sign; the host's edges are convex, since a concave one would demand a reflex angle of the holed face at an outer vertex or of a triangle. This gives the glued and the dented tetrahedron. (b) x sees $a, b \in C_1$ and $d \in C_2$, y sees $c \in C_1$ and $e, f \in C_2$ (the graph is C6): the face containing c–a and c–y continues at a along a–x and at y to e or f, so it contains a, c and a hole vertex, three non-collinear points of P (a hole vertex is strictly inside the outer triangle), and $x \in P$, a contradiction. Finally a disconnected skeleton with disc faces on at most eight vertices is $K_4 \oplus K_4$ on two spheres; each is a tetrahedron by Lemma 4, hence all-convex or all-concave, and Lemma 3 forbids both concave: two tetrahedra, or a tetrahedron with a cavity. Together with Section 4 this proves Theorem 1.

7 The smallest nonplanar skeleton

Knuth's fifth question asks for the smallest realizable nonplanar skeleton and conjectures fourteen vertices, realized by the Szilassi polyhedron [13]. Corollary 5 gives ten as a lower bound; two more lemmas raise it to twelve.

Lemma 9 (line parity)**.** *Let S be a component of ∂X such that every vertex of S lies on exactly three face planes. For two face planes P, P' the number of vertices of S on the line $\ell = P \cap P'$ is even.*

Proof. A vertex v on ℓ lies on the planes P, P', P'' of its faces (F3) and on a unique face $F \subset P$ (two faces at v are adjacent, hence not coplanar). The two sides of F at v lie on the lines $P \cap P'$ and $P \cap P''$, so exactly one side of F at v lies along ℓ; such a side is an edge of S, a segment between consecutive vertices on ℓ with no vertex in its interior (F2). Thus the sides along ℓ pair the vertices on ℓ. □

Lemma 10 (collinear boundary edges of a holed face)**.** *Let Φ be a face in plane P with one hole, its outer boundary and hole boundary being a triangle or a quadrilateral each. Then no* *two edges of Φ are collinear, except that when the outer boundary is a dart (a quadrilateral with a reflex vertex r) hole edges may lie on the lines of the two outer edges at r, beyond r; if the hole is a triangle, at most one hole edge does. Every other face meets P in a line, so the edges of Φ it contains are collinear.*

Proof. Two edges of a simple triangle or quadrilateral are never collinear. A point strictly inside a convex polygon lies strictly on one side of each edge line, so a convex outer boundary admits no collinear hole edge. For a dart, only the lines of the two edges at r enter the interior, and they meet at r; two hole edges on them would make r a hole vertex. □

Theorem 11. *No surface component of a 3VP with ten vertices carries a nonplanar skeleton, connected or not. Hence every realizable nonplanar signed skeleton has at least twelve vertices.*

Proof. Disc faces. By Corollary 5 the component lies on a torus with $F = 5$ faces in $p \leq 5$ planes. By F3 each vertex is the intersection point of the three distinct planes of its faces and distinct vertices give distinct triples, so $10 = V \leq \binom{p}{3} \leq 10$: $p = 5$, every triple of planes meets in one point, the ten points are distinct and all are vertices, no four planes are concurrent, and each plane carries one face. The line $P_i \cap P_j$ then contains the three vertices $P_i \cap P_j \cap P_k$, an odd number, contradicting Lemma 9.

Holed faces. A holed face $\Phi \subset P$ has vertex-disjoint boundary cycles in P, each vertex of which has a neighbour off P. Two holes would need nine coplanar vertices served by a single off-plane vertex of degree three. A machine enumeration over the nonplanar cubic graphs on ten vertices, the disconnected $K_{3,3} \oplus K_4$ included, shows that the one-hole configurations have cycle lengths $(3, 4)$ or $(3, 3)$. Two holed faces $\Phi_1 \subset P_1$, $\Phi_2 \subset P_2$ share at least two boundary vertices; $P_1 = P_2$ would make two faces at a shared vertex coplanar and adjacent; otherwise the shared vertices lie on $\ell = P_1 \cap P_2$ and, as in Lemma 9, are perfectly matched by edges on ℓ common to both boundaries, with each cycle contributing at most one edge and two edges of one face only as in Lemma 10. Enumerating the pairs of configurations that satisfy these conditions and the off-plane neighbour conditions leaves 220 labelled configurations of a single shape: both faces have a quadrilateral outer boundary and a triangular hole and share one quadrilateral edge and one triangle edge on ℓ; Lemma 10 then makes both quadrilaterals darts whose reflex vertex is the endpoint of the shared quadrilateral edge nearer the hole segment, the same vertex for both faces, contradicting Lemma 2 (a vertex has at most one reflex face). So exactly one face has a hole. The skeleton is nonplanar, so the surface has genus $g \geq 1$, and $F = V/2 + 2 - 2g + 1 = 8 - 2g \geq 4$ gives $g \in \{1, 2\}$. If $g = 2$ then $F = 4$, the component lies in exactly four planes and is a tetrahedron by Lemma 4, which has no hole; so $g = 1$, $F = 6$, and there are five faces other than Φ in at most $p - 1 \leq 5$ planes, with $p \geq 5$. If $p = 5$, the disc-face argument applies verbatim (all ten vertices are the triple points of five planes in general position) and Lemma 9 is violated. If $p = 6$, the other faces meet P in at most five lines, and by Lemma 10 at most one line carries two edges of Φ: the six or seven edges of Φ need six lines. Contradiction. □

Lemma 12 (general position)**.** *If a signed cubic graph is realizable, it is realizable by a 3VP whose face planes are in general position: no two parallel, no three through a line, no four through a point, and no vertex on a face plane other than the three of its faces.*

Proof. Perturb the normals and offsets of all face planes by a small ε, redefine each vertex as the intersection of the perturbed planes of its three faces (a single point for small ε by F3), and each face as the polygon in its perturbed plane with the same corner sequences. Adjacent faces share the planes of the vertices of their common edge, so perturbed edges lie on intersection lines and every corner of a perturbed face lies in its plane. Simplicity of the face polygons, holes strictly inside, the absence of 180° angles, the strict signs of dihedral angles, positive enclosed volume, and faces meeting only along shared edges and vertices are strict inequalities or positive distances at $\varepsilon = 0$ (two faces in distinct planes meet only on the line of the planes, where each occupies finitely many closed intervals with endpoints at side crossings; at $\varepsilon = 0$ the intervals of non-adjacent faces are disjoint and those of adjacent faces overlap exactly in the shared edge, with positive gaps elsewhere, and the endpoints move continuously), so they persist for small ε; the four general-position conditions are proper algebraic conditions on ε, so they hold for generic ε. □

Lemma 13. *For planes in general position with normals n_i and offsets d_i, let χ be the chirotope of the vectors $v_0 = (0, 0, 0, 1)$ and $v_i = (n_i, -d_i)$. Then χ determines the order of the triple points along every line $P_i \cap P_j$ and the side of every plane containing every triple point; reorienting any v_i (including v_0) or relabelling the planes describes the same arrangement.*

Proof. With $d = n_i \times n_j$ and the triple point with P_k written as $x_0 + s_k d$, expanding $\det[v_i, v_j, v_k, v_l]$ gives $\operatorname{sign}(s_l - s_k) = c\,\chi(i, j, k, l)\,\chi(0, i, j, k)\,\chi(0, i, j, l)$ with c depending only on (i, j) and the parametrization; the side statement is $\chi(i, j, k, l)$ up to a sign depending on (i, j, k). Replacing v_i by $-v_i$ leaves the plane $n_i \cdot x = d_i$ unchanged. □

Theorem 14. *No 3VP whose faces are discs has a nonplanar skeleton component with twelve vertices.*

Proof. By Lemma 12 assume general position. By Corollary 5, $F = 8 - 2g \geq 4$; $g = 2$ would give a tetrahedron by Lemma 4, with four vertices instead of twelve; so the component lies on a torus with six faces in $p \leq 6$ planes, and $12 = V \leq \binom{p}{3}$ forces $p = 6$, one face per plane, the vertices being twelve of the twenty triple points. By Lemma 13 the following search depends only on the chirotope up to relabelling and reorientation. Enumerating all uniform rank-4 chirotopes [11, 10] on seven elements from the three-term Grassmann–Plücker relations gives 1,743,360 chirotopes with one basis sign fixed and exactly 43 orbits with the element at infinity fixed (mirror images identified); random arrangements, classified by canonical chirotope, exhibit $86 = 2 \cdot 43$ types, saturating on four independent seeds; reflecting the representative of each type shows that no type is achiral and that the mirror map is a fixed-point-free involution on the 86 types, so they are exactly the 43 orbits together with their mirror images and every type is represented. For each type the search enumerates all twelve-subsets W of the triple points with an even number of points on every line (Lemma 9), pairs consecutive points of W along each line into edges, requires the edges in each plane to form a single simple polygon (a crossing test at every triple point of the plane), requires a connected nonplanar skeleton, orients the surface, and checks that faces meet only along shared edges and vertices. The search was run in floating point on the first sample of each type and on three further samples per type, and then repeated in exact rational arithmetic on four integer-coefficient arrangements of each type (two independent seeds, two representatives each), with every combinatorial candidate certified exactly. No twelve-subset passes in any type. □

Theorem 15. *No surface component of a 3VP with twelve vertices carries a nonplanar skeleton, connected or not.*

Proof. Assume general position (Lemma 12); let the surface component carry $V = 12$ vertices and $E = 18$ edges (its skeleton may be disconnected; it is nonplanar iff some connected component of it is), with F faces with H holes, genus g, in p planes. Distinct vertices give distinct triples of planes (F3), so $12 \leq \binom{p}{3}$ and $p \geq 6$. In each of its three planes a vertex lies on exactly one boundary cycle and has exactly two edges, so cycles in a plane are vertex-disjoint and all cycles together have length $3V = 36$; as every cycle has at least three vertices, $C = F + H \leq 12$. Every plane carries a face, so $F \geq p$. Euler gives $F - H = 8 - 2g$, which is even, so C is even and $F = (C + F - H)/2 \leq 9$; thus $6 \leq p \leq F \leq 9$.

By Lemma 9 each line carries an even number of vertices, which pair into edges. The pairing depends on the arrangement, but allowing *every* perfect matching on every line gives a coordinate-free relaxation: the vertex sets are the twelve-element sets of plane-triples meeting each pair evenly, and the skeletons come from choosing matchings. A survivor must have a nonplanar cubic skeleton (connectivity is not required), vertex-disjoint cycles of length ≥ 3 in each plane, C even with $2p - 6 \leq C \leq 12$, and a consistent orientation of all cycles.

For $p = 7$ and $p = 8$ the vertex sets number 48,020 and 345,100 (computed twice, as the kernel of the incidence matrix over $\mathbf{F}_2$ and by backtracking), giving 284,970 and 587,860 configurations; none survives. For $p = 9$, $F = 9$ forces $C = 12$, $H = 3$, $g = 1$ and all twelve cycles triangular. A vertex v with planes P, Q, R has one edge on each of the three lines through it, and the cycle of plane P through v uses exactly the two edges of v lying in P; so the three cycles at v use the three distinct pairs of its edges, and each being a triangle forces the two neighbours in each pair to be adjacent. Hence the three neighbours of v are pairwise adjacent and v with them induces K_4, which in a cubic graph is a whole component. The skeleton is a disjoint union of K_4's, hence planar, a contradiction. For $p = 6$ every survivor has $C = 6$, six faces without holes, one to a plane, so by Corollary 5 the skeleton on the surface is connected, as the search of Theorem 14 requires, and that theorem applies. Finally, run at fourteen vertices on seven planes with the Euler parity adjusted, the same enumeration and filters accept the Szilassi polyhedron, so they do not reject genuine solids. □

Theorem 16. *The smallest realizable nonplanar signed skeleton has exactly fourteen vertices, attained by the Szilassi polyhedron.*

Proof. The skeleton is the disjoint union of the skeletons on the surface components of ∂X, so it is nonplanar iff the skeleton on some surface component S is, and X has at least as many vertices as S; that number is even. Six and eight vertices are excluded by Corollary 5 (there is no disconnected cubic graph on six vertices, and the only one on eight, $K_4 \oplus K_4$, is planar), ten by Theorem 11 and twelve by Theorem 15, in each case for any skeleton on S, connected or not; the Szilassi polyhedron realizes fourteen. □

8 Remarks and open problems

Of Knuth's five problems, Peter Weigel answered the third and fourth in 2023 (integer coordinates always exist; dihedral angles need not be multiples of 30°), as Knuth's note records. The first, a decision procedure for realizability, and the second, the classification of the realizations of one signature up to homotopy, remain open; so does his question whether the Heawood graph carries any signature other than the Szilassi one. Our lemmas leave 868 candidate classes; a numerical search over all of them in every labelling relative to the unique torus face structure, validated by recovering the Szilassi solid itself and escalated with penalties for crossing faces, realizes only the Szilassi signature, with no other class within 10^{-3} of the minimum. We take this as evidence, not proof, that there is no other. At ten vertices, which he expected to be large, the lemmas leave 477 sign patterns on the eight planar cubic graphs; a realizer seeded from a spherical Tutte layout has certified 240 of them with exact rational coordinates (all five all-convex ones among them, as Steinitz's theorem requires), and the remaining 237 are undecided, since the lemmas are not known to be tight at ten vertices. We also built the catalogue of the 40,016 combinatorial types of seven planes in general position, by single-element extension of the 43 types of six planes, with non-realizable candidates excluded by biquadratic final polynomials [12], for a search of the holed-face case that was superseded by the coordinate-free argument of Theorem 15 before it completed; it plays no part in the proofs.

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References

[1] D. E. Knuth, Signed skeletons, unpublished note, April 2020, revised August 2026. `https://cs.stanford.edu/~knuth/papers/signed-skeletons.pdf`

[2] D. A. Huffman, Impossible objects as nonsense sentences, *Machine Intelligence* 6 (1971), 295–323.

[3] M. B. Clowes, On seeing things, *Artificial Intelligence* 2 (1971), 79–116.

[4] K. Sugihara, *Machine Interpretation of Line Drawings*, MIT Press, 1986.

[5] W. Whiteley, Realizability of polyhedra, *Structural Topology* 1 (1979), 46–58.

[6] D. Eppstein and E. Mumford, Steinitz theorems for simple orthogonal polyhedra, *Journal of Computational Geometry* 5 (2014), 179–244.

[7] E. Steinitz, *Polyeder und Raumeinteilungen*, Encyklopädie der mathematischen Wissenschaften III.1.2, Teubner, 1922.

[8] H. Tietze, Über Konvexheit im kleinen und im großen und über gewisse den Punkten einer Menge zugeordnete Dimensionszahlen, *Math. Z.* 28 (1928), 697–707.

[9] S. Nakajima, Über konvexe Kurven und Flächen, *Tôhoku Math. J.* 29 (1928), 227–230.

[10] L. Finschi and K. Fukuda, Generation of oriented matroids: a graph theoretical approach, *Discrete Comput. Geom.* 27 (2002), 117–136.

[11] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler, *Oriented Matroids*, 2nd ed., Cambridge University Press, 1999.

[12] J. Bokowski and J. Richter-Gebert, On the finding of final polynomials, *European J. Combin.* 11 (1990), 21–34.

[13] L. Szilassi, Regular toroids, *Structural Topology* 13 (1986), 69–80.

[14] R. C. Read and R. J. Wilson, *An Atlas of Graphs*, Oxford University Press, 1998.

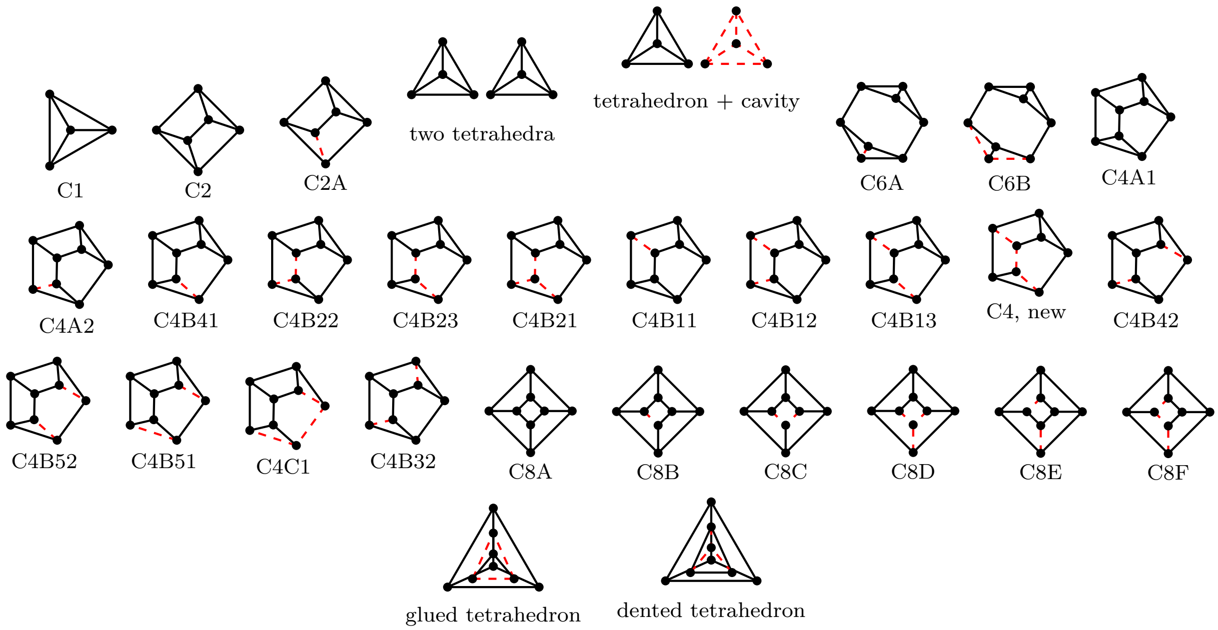

Figure 1: The thirty realizable signatures on at most eight vertices (solid: convex, dashed: concave), each labelled with Knuth's model name or "new".

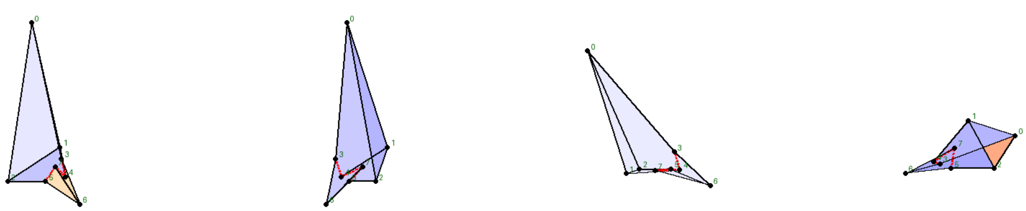

Figure 2: Four views of the new C4 signature. Faces $\{1,0,2\}$, $\{2,0,3,6,5\}$, $\{3,0,1,4\}$, $\{4,1,2,5,7\}$, $\{6,3,4,7\}$, $\{7,5,6\}$; vertices $0=(-6,72,-36)$, $1=(-6,16,20)$, $2=(-27,-12,6)$, $3=(12,0,0)$, $4=(12,-8,8)$, $5=(0,-12,6)$, $6=(24,-24,12)$, $7=(0,0,12)$; edges 3–4, 4–7, 5–7 concave (dashed), all others convex; vertex types $0,0,0,0,1,1,2,2$.